

Innerness of Derivations on Subalgebras of Measurable Operators

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Abstract

Given a von Neumann algebra M with a faithful normal semi-finite trace τ , let $L(M, \tau)$ be the algebra of all τ -measurable operators affiliated with M . We prove that if A is a locally convex reflexive complete metrizable solid $*$ -subalgebra in $L(M, \tau)$, which can be embedded into a locally bounded weak Fréchet M -bimodule, then any derivation on A is inner.

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1. Introduction

The structure of automorphisms and derivations of operator algebras is an important part of the theory of operator algebras and their applications in quantum dynamics.

Recall that a linear operator D on an algebra A is called a *derivation* if it satisfies the condition

$$D(xy) = D(x)y + xD(y)$$

for all $x, y \in A$.

Every (but fixed) element $a \in A$ generates a derivation D_a on A , defined as $D_a(x) = ax - xa$, $x \in A$. Such derivations are said to be *inner* derivations.

Derivations on C^* -algebras and von Neumann algebras have been studied in the monographs of Sakai [13], [14]. It is well-known that every derivation on a C^* -algebra A is norm continuous and if C^* -algebra A is unital and simple or it is weakly closed (i.e. is a von Neumann algebra) then any derivation on A is inner. For general Banach algebras similar problems were considered in the monograph [8].

Investigation of derivations on unbounded operator algebras and, in particular, on the algebra $L(M)$ of measurable operators affiliated with a von Neumann algebra M , was initiated in the papers [5], [6]. One of the main problems posed in these papers was: whether any derivation on $L(M)$ is inner. A negative answer to this problem in the general setting was given in the paper [7] (see also [10]). Namely, it was proved that if M is a non atomic abelian von Neumann algebra (in particular $L^\infty(0; 1)$) then $L(M)$ (resp. $L^0(0; 1)$) admits a non trivial (and hence discontinuous, and non inner) derivation.

Further there were some positive results on this way. In [3] we have proved that if M is type I von Neumann algebra with a faithful normal semi-finite trace τ , then a derivation on the algebra $L(M, \tau)$ of all τ -measurable operators affiliated with M is inner if and only if it is Z -linear, or equivalently if it is identically zero on the center Z of M . Recently [4] we gave a complete description of derivation on $L(M, \tau)$ and in particular proved that if M is of type I_∞ then any derivation on $L(M, \tau)$ is inner.

If M is a general von Neumann algebra with a faithful normal semi-finite trace τ , then the algebra $L(M, \tau)$ contains various subalgebras with different properties of derivations. One of the interesting classes of subalgebras in $L(M, \tau)$ are so called Arens algebras

$$L^\omega(M, \tau) = \bigcap_{p \geq 1} L^p(M, \tau).$$

In the paper [2] we gave a complete description of derivations on $L^\omega(M, \tau)$ and proved that any derivation on $L^\omega(M, \tau)$ is inner if and only if the trace τ is finite.

In this connection a natural question arises:

which subalgebras in $L(M, \tau)$ admit only inner derivation?

In this paper we give a sufficient condition for subalgebras in $L(M, \tau)$ to have such a property. Namely, we prove that if A is locally convex reflexive complete metrizable solid $*$ -subalgebra in $L(M, \tau)$, which can be embedded into a locally bounded weak Fréchet M -bimodule, then any derivation on A is inner.

2. Preliminaries

Let H be a Hilbert space, and let $B(H)$ be the algebra of all bounded linear operators on H . Consider a von Neumann algebra $M \subset B(H)$ with a faithful normal semi-finite trace τ , and denote by $\mathcal{P}(M)$ the lattice of (orthogonal) projections in M .

A linear subspace \mathcal{D} in H is affiliated with M (denoted as $\mathcal{D}\eta M$), if $u(\mathcal{D}) \subset \mathcal{D}$ for any unitary operator u from the commutant

$$M' = \{y \in B(H) : xy = yx, \forall x \in M\}$$

of the von Neumann algebra M .

A linear operator x with the domain $\mathcal{D}(x) \subset H$ is said to be affiliated with M (denoted as $x\eta M$) if $u(\mathcal{D}(x)) \subset \mathcal{D}(x)$ and $ux(\xi) = xu(\xi)$ for all $u \in M'$, $\xi \in \mathcal{D}(x)$.

A linear subspace \mathcal{D} in H is called τ -dense, if

- 1) $\mathcal{D}\eta M$;
- 2) given any $\varepsilon > 0$ there exists a projection $p \in \mathcal{P}(M)$ such that $p(H) \subset \mathcal{D}$ and $\tau(p^\perp) \leq \varepsilon$.

A closed linear operator x is called τ -measurable with respect to the von Neumann algebra M , if $x\eta M$ and $\mathcal{D}(x)$ is τ -dense in H .

Denote by $L(M, \tau)$ the set of all τ -measurable operators affiliated with M . Consider the topology t_τ of convergence in measure on $L(M, \tau)$, which is defined by the following neighborhoods of zero:

$$V(\varepsilon, \delta) = \{x \in L(M, \tau) : \exists e \in \mathcal{P}(M), \tau(e^\perp) \leq \delta, xe \in M, \|xe\|_M \leq \varepsilon\},$$

where $\|\cdot\|_\infty$ is the operator norm on M , and ε, δ are positive numbers.

It is well-known [12] that $L(M, \tau)$ equipped with the measure topology is a complete metrizable topological $*$ -algebra.

Now let us recall the notion of a bimodule over a Banach algebra (see [8]).

Let A be a complex algebra and let E be a complex linear space. E is called a left A -module (respectively right A -module) if a bilinear map $(a, x) \mapsto a \cdot x$ (respectively $(a, x) \mapsto x \cdot a$) from $A \times E$ into E is defined, such that given any $a, b \in A$ and $x \in E$ one has

$$a \cdot (b \cdot x) = ab \cdot x \quad (\text{respectively } (x \cdot a) \cdot b = x \cdot ab),$$

E is said to be A -bimodule if is left and right A -module simultaneously, and

$$a \cdot (x \cdot b) = (a \cdot x) \cdot b,$$

for all $a, b \in A, x \in E$.

Let A be a Banach algebra and suppose that E is a Fréchet space, i.e. a complete metric space with a shift invariant metric. If E is an A -bimodule and the maps $x \mapsto a \cdot x$ and $x \mapsto x \cdot a$ are continuous for each $a \in A$, then E is called a *weak Fréchet* A -bimodule.

Interesting examples of weak Fréchet A -bimodules are given by non commutative L^p -spaces $L^p(M, \tau) \subset L(M, \tau)$, $p \geq 1$. Indeed, given any $a \in M$ and $x \in L^p(M, \tau)$ one has $ax \in L^p(M, \tau)$, $xa \in L^p(M, \tau)$ and $\|ax\|_p \leq \|a\|_\infty \|x\|_p$ which imply the above statement.

Let E and F be metrizable linear topological spaces and let $T : E \rightarrow F$ be a linear operator. The *separating space* of the linear map T , denoted by $\mathcal{S}(T)$, is defined as

$$\mathcal{S}(T) = \{y \in F : \text{there is } (x_n)_{n \in \mathbb{N}} \text{ in } E \text{ such that } x_n \rightarrow 0 \text{ and } T(x_n) \rightarrow y\}.$$

Recall that $\mathcal{S}(T)$ is closed (see [8], Proposition 5.1.2) and that the closed graph theorem is valid for complete metrizable topological linear space. Therefore T is continuous if and only if $\mathcal{S}(T) = \{0\}$.

Definition 2.1. [8]. Let A be an algebra, and let E be a topological linear space which is a A -bimodule. Then E is a *separating module* if, for each sequence $\{a_n\}$ in A , the nest $\overline{(a_1 \cdots a_n E)}$ stabilizes, i.e. there is a $n_0 \in \mathbb{N}$ such that $\overline{(a_1 \cdots a_n E)} = \overline{(a_1 \cdots a_{n+1} E)}$ for all $n > n_0$, where \overline{F} – the closure of the set F .

A linear topological space E is said to be *locally bounded* if there exists a bounded neighborhood of zero in E .

From [8, Theorem 5.2.15], we have the following

Proposition 2.2. *Let E be a weak Fréchet A -bimodule, and let $D : A \rightarrow E$ be a derivation. Then*

- 1) *The separating space $\mathcal{S}(D)$ is a closed submodule of E ;*
- 2) *Suppose that E is locally bounded. Then $\mathcal{S}(D)$ is a separating module.*

Given a linear topological space X with a topology t_X , let us denote by $x_n \xrightarrow{t_X} 0$ the convergence in the topology t_X .

If (A, t_A) and (B, t_B) are linear topological spaces, with $A \subseteq B \subseteq L(M, \tau)$ then we shall suppose that the topology t_A is stronger than t_B , i.e. (A, t_A) is topologically imbedded into (B, t_B) .

3. The main results

The aim of the present section is to prove the following result.

Theorem 3.1 *Let M be a von Neumann algebra with a faithful normal semi-finite trace τ . Suppose that A is a complete metrizable solid $*$ -subalgebra in $L(M, \tau)$ and E is a locally bounded weak Fréchet M -bimodule in $L(M, \tau)$. If*

- 1) A is locally convex and reflexive;
 - 2) $M \subset A \subset E$ are topological imbedding,
- then any derivation of the algebra A is inner.*

Recall that a subalgebra A in $L(M, \tau)$ is solid if $x \in A, y \in L(M, \tau), |y| \leq |x|$ implies $y \in A$.

The proof of this theorem consists of several steps.

Proposition 3.2. *Given an arbitrary von Neumann algebra M , and a weak Fréchet M -bimodule E , suppose that p is a projection in M and $D : M \rightarrow E$ is a derivation, i.e. a linear map such that $D(xy) = D(x)y + xD(y)$ for all $x, y \in M$. Put $D_p(x) = pD(x)p, x \in pMp$. Then*

- 1) $D_p : pMp \rightarrow pEp$ is a derivation;
- 2) $p\mathcal{S}(D)p \subseteq \mathcal{S}(D_p)$.

Proof. 1) For $x, y \in pMp$ we have $x = pxp, y = pyp$. Therefore $D_p(xy) = pD(pxy)p = pD(pxpyp)p = pD(pxp)pyp + pxpD(pyp)p = D_p(x)y + xD_p(y)$, i.e. D_p is a derivation.

2) For $y \in \mathcal{S}(D)$ according the definition there exists a sequence $\{x_n\}$ in M such that $x_n \xrightarrow{\|\cdot\|} 0$ and $D(x_n) \xrightarrow{t_E} y$ as $n \rightarrow \infty$. But then $px_np \xrightarrow{\|\cdot\|} 0$ and $D_p(px_np) = pD(px_np)p = pD(p)x_np + pD(x_n)p + px_nD(p)p \xrightarrow{t_E} pyp$, i.e. $D_p(px_np) \xrightarrow{t_E} pyp$, which means that $pyp \in \mathcal{S}(D_p)$ and therefore $p\mathcal{S}(D)p \subseteq \mathcal{S}(D_p)$. The proof is complete. ■

Proposition 3.3. *Let M be a von Neumann algebra with a faithful normal semi-finite trace τ and let E be a locally bounded weak Fréchet M -bimodule in $L(M, \tau)$. Then every derivation $D : M \rightarrow E$ is automatically continuous.*

Proof. Let us show that $\mathcal{S}(D) = \{0\}$. Suppose the opposite, i.e. $\mathcal{S}(D) \neq \{0\}$ and take a non zero $y \in \mathcal{S}(D)$. Chose a projection e in M such that $ye \in M$ and $ye \neq 0$. Since $\mathcal{S}(D)$ is a submodule in E , we have that $ye(ye)^* \in \mathcal{S}(D)$. Thus without loss of the generality we may suppose that $y \geq 0$. Take a projection $p \in M$ such that $pyp \neq 0$ and $n^{-1}p \leq pyp \leq np$ for an appropriate $n \in \mathbb{N}$. Then pyp is invertible in pMp , i.e.

there exists an element $z \in pMp$ such that $pypz = p$. Since $\mathcal{S}(D)$ is an M -bimodule it follows that $p \in \mathcal{S}(D)$. Now consider two cases separately:

The case 1. pMp is finite dimensional. Observe the derivation $D_p : pAp \rightarrow pEp$ defined by

$$D_p(x) = pD(pxp)p, \quad x \in pAp.$$

Since pMp is finite dimensional, the spaces pAp and pEp are also finite dimensional as subspaces of $L(M, \tau) = pMp$. Therefore D_p is necessary continuous, i.e. $\mathcal{S}(D_p) = \{0\}$.

On the other hand Proposition 3.2 implies that $p\mathcal{S}(D)p \subset \mathcal{S}(D_p)$, and from the consideration above we have that $p \in \mathcal{S}(D_p)$ and hence $0 \neq p \in p\mathcal{S}(D)p \subset \mathcal{S}(D_p) = \{0\}$. This contradiction implies that $\mathcal{S}(D) = \{0\}$.

The case 2. pMp is infinite dimensional. In this case there exists a strictly monotone decreasing sequence (p_n) of projections in M such that $p_n \leq p$ for all $n \in \mathbb{N}$. Since $\mathcal{S}(D)$ is a closed submodule in E (Proposition 2.2) $p_n\mathcal{S}(D)$ is also closed. Further $p_n \neq p_{n+1} \leq p$ implies that

$$\overline{p_n\mathcal{S}(D)} \neq \overline{p_{n+1}\mathcal{S}(D)}. \quad (1)$$

On the other hand, since E is locally bounded Proposition 2.2 implies that the nest $\{\overline{p_n\mathcal{S}(D)}\}$ stabilizes, i.e. there exist $n_0 \in \mathbb{N}$ such that $\overline{p_n\mathcal{S}(D)} = \overline{p_{n+1}\mathcal{S}(D)}$ for all $n > n_0$ in a contradiction with (1). Therefore $\mathcal{S}(D) = \{0\}$ and the proof is complete.

Remark 1. Proposition 3.3 implies the well-known fact that every derivation on the von Neumann algebra M is norm continuous. It is sufficient to put $E = M$.

Proposition 3.4. *Let M be a von Neumann algebra with a faithful normal semi-finite trace τ . Suppose that E is a locally bounded weak Fréchet M -bimodule in $L(M, \tau)$ and A is complete metrizable algebra such that $A \subseteq E$. Then any derivation $D : M \rightarrow A$ is continuous.*

Proof. Let $\{x_n\} \subset M, x_n \xrightarrow{\|\cdot\|} 0$ and $D(x_n) \xrightarrow{t_A} y$, which implies that $D(x_n) \xrightarrow{t_\tau} y$. Let us show that $y = 0$. By Proposition 3.3 $D : A \rightarrow E$ is continuous and thus $D(x_n) \xrightarrow{t_E} 0$ and hence $D(x_n) \xrightarrow{t_\tau} 0$. This implies that $y = 0$. The proof is complete. ■

If A is a locally convex metrizable space, then its topology can be generated by an increasing sequence of seminorms $\{\rho_n, n \in \mathbb{N}\}$.

Denote by U the group of all unitaries of the von Neumann algebra M .

Proposition 3.5. *Let A be a locally convex weak Fréchet M -bimodule in $L(M, \tau)$. Given any non zero element $x \in A$ there exists a seminorm ρ_n such that*

$$\inf\{\rho_n(uxu^*) : u \in U\} \neq 0.$$

Proof. Suppose that opposite, i.e. $\inf\{\rho_n(uxu^*) : u \in U\} = 0$ for all $n \in \mathbb{N}$. Chose the unitaries $u_n \in U$ such that $\rho_n(u_nxu_n^*) \leq n^{-1}$. Since $\rho_k \leq \rho_{k+1}$, we have that $\rho_k(u_nxu_n^*) \rightarrow 0$ as $n \rightarrow \infty$ for each fixed $k \in \mathbb{N}$. This means that $u_nxu_n^* \xrightarrow{t_A} 0$, and hence $u_nxu_n^* \xrightarrow{t_\tau} 0$. Since $\|u_n\|_\infty = 1$ for all $n = 1, 2, \dots$, we obtain that $x = u_n^*(u_nxu_n^*)u_n \xrightarrow{t_\tau} 0$, i.e. $x = 0$ a contradiction. The proof is complete. ■

Proposition 3.6. *Let M , A and E be as in theorem 3.1. Then every derivation $D : M \rightarrow A$ is spatial, i.e. $D(x) = ax - xa$ for an appropriate $a \in A$ and every $x \in M$.*

Proof. By Proposition 3.4 the derivation $D : M \rightarrow A$ is continuous.

Let U be the group of all unitary elements in M . Given any $u \in U$ put

$$T_u(x) = uxu^* + D(u)u^*, \quad x \in A.$$

Since map $x \mapsto uxu^*$ is continuous, the map T_u is $\sigma(A, A^*)$ -continuous.

For $u, v \in U$ we have

$$\begin{aligned} T_u(T_v(x)) &= T_u(vxv^* + D(v)v^*) = u(vxv^* + D(v)v^*)u^* + D(u)u^* = uvxv^*u^* + \\ &u D(v)v^*u^* + D(u)u^* = (uvx + D(u)v + uD(v))(uv)^* = uvx(uv)^* + D(uv)(uv)^* = T_{uv}(x), \end{aligned}$$

i. e.

$$T_u T_v = T_{uv}, \quad u, v \in U. \quad (2)$$

Further we have $T_u(0) = D(u)u^*$ and the continuity of D implies that the set $K_D = \{T_v(0) : v \in U\} = \{D(v)v^* : v \in U\}$ is bounded in A . Moreover, the set $K = \text{cl}(\text{co}(K_D))$ – the closure of the convex hull of K_D , is a closed convex bounded subset in A . The reflexivity of the space A then implies that K is a non-void $\sigma(A, A^*)$ -compact convex set. From (2) it follows that $T_u(K_D) \subset K_D$ for all $u \in U$. Since T_u is an affine homeomorphism we have $T_u(\text{cl}(\text{co}(K_D))) = \text{cl}(\text{co}(T_u(K_D))) \subset \text{cl}(\text{co}(K_D))$, i. e. $T_u(K) \subseteq K$ for all $u \in U$.

According Proposition 3.6, given any $x, y \in A, x \neq y$ there exists seminorm ρ_n such that

$$\inf\{\rho_n(T_u(x) - T_u(y)) : u \in U\} = \inf\{\rho_n(u(x - y)u^*) : u \in U\} \neq 0.$$

Therefore $\{T_u : u \in U\}$ is a non-contracting (in the sense of [11]) semigroup of $\sigma(A, A^*)$ -continuous affine mappings of a $\sigma(A, A^*)$ -compact convex set K . By Ryll-Nardzewski's fixed point theorem [11], there exists $a \in K$ such that $T_u(a) = a$ for all $u \in U$. This means that $uau^* + D(u)u^* = a$, i. e. $D(u) = au - ua$ for all $u \in U$. Since every element of M is a linear combination of unitaries from M , we have $D(x) = ax - xa$ for all $x \in M$, i.e. D is a spatial derivation on M with values in A . The proof is complete. ■

Proof of Theorem 3.1. According to Proposition 3.6 there exists element $a \in A$ such that $D(x) = ax - xa$ for all $x \in M$. We shall proof that this is true for all $x \in A$.

First suppose that $x \in A$, $x \geq 0$. In this case the element $\mathbf{1} + x \in A \subset L(M, \tau)$ is invertible and moreover $(\mathbf{1} + x)^{-1} \in M$.

For an invertible element $b \geq 0$ in A one has $0 = D(\mathbf{1}) = D(bb^{-1}) = D(b)b^{-1} + bD(b^{-1})$, i. e. $D(b) = -bD(b^{-1})b$.

Therefore

$$D(x) = D(\mathbf{1} + x) = -(\mathbf{1} + x)D((\mathbf{1} + x)^{-1})(\mathbf{1} + x).$$

On the other hand, since $(\mathbf{1} + x)^{-1} \in M$ we have from the above

$$D((\mathbf{1} + x)^{-1}) = a(\mathbf{1} + x)^{-1} - (\mathbf{1} + x)^{-1}a.$$

Therefore,

$$\begin{aligned} -(\mathbf{1} + x)D((\mathbf{1} + x)^{-1})(\mathbf{1} + x) &= -(\mathbf{1} + x)[a(\mathbf{1} + x)^{-1} - \\ & - (\mathbf{1} + x)^{-1}a](\mathbf{1} + x) = -(\mathbf{1} + x)a + a(\mathbf{1} + x) = ax - xa, \end{aligned}$$

i. e.

$$D(x) = -(\mathbf{1} + x)D((\mathbf{1} + x)^{-1})(\mathbf{1} + x) = ax - xa.$$

Therefore $D(x) = ax - xa$ for every $x \in A$, $x \geq 0$. Since A is a solid $*$ -subalgebra in $L(M, \tau)$, every element from A is a linear combination of positive elements of A . Thus $D(x) = ax - xa$ for all $x \in A$. The proof is complete. ■

Remark 2. The condition on A to be solid is used only at the end of proof of Theorem 3.1. In fact this condition may be replaced by the condition $Lin(A_+) = A$, where A_+ is the positive cone of A and $Lin(A_+)$ is the linear span of A_+ (i.e. the positive cone A_+ is hereditary in A).

Example 3.8. An example of algebras, satisfying the conditions of Theorem 3.1 is given by a non commutative Arens algebra $L^\omega(M, \tau)$ in the case of a finite trace τ (see [2]).

Given be a von Neumann algebra M with a faithful normal semi-finite trace τ M and $p \geq 1$, put $L^p(M, \tau) = \{x \in L(M, \tau) : \tau(|x|^p) < \infty\}$. It is known [12] that $L^p(M, \tau)$ is a Banach space with respect to the norm

$$\|x\|_p = (\tau(|x|^p))^{1/p}, \quad x \in L^p(M, \tau).$$

Consider the space

$$L^\omega(M, \tau) = \bigcap_{p \geq 1} L^p(M, \tau).$$

It is known [1], [9] that $L^\omega(M, \tau)$ is a locally convex metrizable $*$ -algebra with the topology t generated by the sequence of norms

$$\|x\|'_n = \max\{\|x\|_1, \|x\|_n\}, \quad n \in \mathbb{N}.$$

The algebra $L^\omega(M, \tau)$ is called a (non commutative) *Arens algebra*. The dual space for $(L^\omega(M, \tau), t)$ was described in [1], where it has been proved that $(L^\omega(M, \tau), t)$ is reflexive if and only if trace τ is finite.

Therefore Theorem 3.1 implies that if the trace τ is finite, then every derivation on the algebra $L^\omega(M, \tau)$ is inner.

It should be noted also that a complete description of derivations on general $L^\omega(M, \tau)$ was obtain in [2]. Namely. it has been proved that every derivation on $L^\omega(M, \tau)$ is spatial and has the form

$$D(x) = ax - xa, \quad x \in L^\omega(M, \tau)$$

for an appropriate $a \in M + L_2^\omega(M, \tau)$, where $L_2^\omega(M, \tau) = \bigcap_{p \geq 2} L^p(M, \tau)$.

Now let us consider an example of an algebra A satisfying all conditions of Theorem 3.1 except $M \subset A$, which admits non-inner derivations.

Example 3.9. Put

$$A = L_2^\omega(M, \tau) = \bigcap_{p \geq 2} L^p(M, \tau)$$

and consider on A the topology generated by the system of norms $\{\|\cdot\|_{p \geq 2}\}$. Then A is a metrizable locally convex $*$ -algebra [2] and

$$L_2^\omega(M, \tau) = \bigcap_{n=2}^{\infty} L^n(M, \tau).$$

As the intersection of countable family of reflexive Banach space, A is also reflexive. If the trace τ is semi-finite but not finite, M is not contained in $A = L_2^\omega(M, \tau)$. Every derivation of $L_2^\omega(M, \tau)$ has the form

$$D_a(x) = ax - xa, \quad x \in L_2^\omega(M, \tau)$$

for an appropriate $a \in M + L_2^\omega(M, \tau)$ (see [2]).

Now if M is non commutative and a is a non central element from $M \setminus L_2^\omega(M, \tau)$ the spatial derivation D_a on $L_2^\omega(M, \tau)$ is not inner.

Let us consider an example of an algebra A which shows that the reflexivity of A is not a necessary condition for the statement of Theorem 3.1.

Example 3.10. Let M be the C^* -product of von Neumann algebra M_n , i.e.

$$M = \bigoplus_{n=1}^{\infty} M_n = \{\{x_n\} : x_n \in M_n, \sup \|x\|_{M_n} < \infty\},$$

where $\|\cdot\|_{M_n}$ is the C^* -norm on M_n .

Put

$$A = \{\{x_n\} : x_n \in M_n, \sum_{n=1}^{\infty} \frac{1}{2^n} \|x_n\|_{M_n}^p < \infty, 1 \leq p < \infty\}.$$

Then it is clear that $\{x_n\} \in A$ if and only if $\{\|x_n\|_{M_n}\} \in l^\omega$, where $l^\omega = \bigcap_{p \geq 1} l^p$ – the Arens algebra associated with abelian von Neumann algebra l^∞ of all bounded complex sequences with the trace

$$\nu(\{\lambda_n\}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \lambda_n, \{\lambda_n\} \in l^\infty.$$

Consider the topology t on A , generated by the family of norms

$$\|x\|_{A,p} = \left(\sum_{n=1}^{\infty} \frac{1}{2^n} \|x_n\|_{M_n}^p \right)^{\frac{1}{p}}.$$

With the coordinatewise algebraic operations and involution (A, t) becomes a locally convex complete metrizable $*$ -algebra. If at least one of the algebras M_n is infinite dimensional, the A is not isomorphic to any Arens algebra.

Consider a derivation $D : A \rightarrow A$. Let q_n be central projection in M such that $q_n M = M_n$. Then we have

$$D(q_n x) = q_n D(x), \quad x \in A,$$

and therefore $D(M_n) \subseteq M_n$ and the restriction

$$D_n(x) = q_n D(x), \quad x \in M_n,$$

gives a derivation $D_n : M_n \rightarrow M_n$. The classical theorem of Sakai implies the existence of an appropriate $a_n \in M_n$ such that $D_n(x) = a_n x - x a_n$ for all $x \in M_n$, moreover one can assume that $\|a_n\| \leq \|D_n\|$ (see [13, Theorem 4.1.6])

Let us show that $a = \{a_n\} \in A$ and that $D(x) = ax - xa$ for all $x \in A$.

Take an arbitrary $\varepsilon > 0$. For each $n \in \mathbb{N}$ there exists $x_n \in M_n$, $\|x_n\|_{M_n} \leq 1$ such that $\|D_n\| \leq \|D_n(x_n)\| + \varepsilon$. Then $x = \{x_n\} \in M$ and $D(x) = \{D_n(x_n)\} \in A$. Therefore $\{\|D(x_n)\|_{M_n}\} \in l^\omega$. The inequalities

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{2^n} \|a_n\|_{M_n}^p &\leq \sum_{n=1}^{\infty} \frac{1}{2^n} \|D_n\|^p \leq \sum_{n=1}^{\infty} \frac{1}{2^n} (\|D_n(x_n)\| + \varepsilon)^p \leq \\ &\leq 2^{p-1} \sum_{n=1}^{\infty} \frac{1}{2^n} (\|D_n(x_n)\|^p + \varepsilon^p) \end{aligned}$$

imply that $\{\|a_n\|_{M_n}\} \in l^\omega$, i.e. $a \in A$. Further since

$$q_n D(x) = D_n(q_n x) = a_n q_n x - q_n x a_n = q_n(ax - xa)$$

for all $n = 1, 2, \dots$, taking the sum over all q_n we obtain $D(x) = ax - xa$ for all $x \in A$. The proof is complete. ■

It well-known that every abelian von Neumann algebra is isomorphic to an algebra $L^\infty(\Omega)$ of all essentially bounded measurable complex functions on a measure space (Ω, Σ, μ) . In this case the algebra τ -measurable operators $L(M, \tau)$ is isomorphic with the algebra $L^0(\Omega)$ of all measurable functions on Ω .

Proposition 3.11. *Let A be a $*$ -subalgebra in $L^0(\Omega)$ and let E be a locally bounded weak Fréchet $L^\infty(\Omega)$ -bimodule such that $L^\infty(\Omega) \subseteq A \subseteq E$. Then every derivation on A is identically zero.*

Proof. Since $L^\infty(\Omega)$ is abelian, every derivation D is equal to zero on idempotents (projections) from $L^\infty(\Omega)$. Therefore $D(x) = 0$ on each step function $x \in A$. The space of step functions is dense in $L^\infty(\Omega)$ and by Proposition 3.4 the derivation $D : L^\infty(\Omega) \rightarrow A$ is continuous, therefore $D(x) = 0$ for all $x \in L^\infty(\Omega)$.

For $x \in A$, take a sequence of idempotents $e_n, n \in \mathbb{N}$ from $L^\infty(\Omega)$ such that $e_n x \in L^\infty(\Omega)$ and $e_n \uparrow \mathbf{1}$. Then $e_n D(x) = D(e_n x) - D(e_n)x = 0$, i.e. $e_n D(x) = 0$ for all $n \in \mathbb{N}$. Since $e_n \uparrow \mathbf{1}$ this implies that $D(x) = 0$. The proof is complete. ■

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